Görtler Instability of Compressible Boundary Layers

Nabil M. El-Hady* and Alok K. Verma†
Old Dominion University, Norfolk, Virginia

Abstract

THE instability of the laminar compressible boundary-layer flows along concave surfaces is investigated. The linearized disturbance equations for the three-dimensional, counter-rotating, longitudinal-type vortices in two-dimensional boundary layers are presented in an orthogonal curvilinear system of coordinates. The basic approximation of the disturbance equations, which includes the effect of the growth of the boundary layer, is considered and solved numerically.

Contents

Centrifugal forces induced by curvature effects lead to the instability of the flow in the form of counter-rotating, vortex-like disturbances (Görtler vortices). These vortices affect indirectly the transition process of boundary-layer flows by modifying the development of unstable waves. Existing compressible linear stability theories neglect the normal velocity component of the basic flow that proved to have a profound effect in reducing the stability of incompressible flows. They only treat the neutral stability case which is of limited value regarding vortex development, possible nonlinear interaction, and transition correlation. Here the amplification of Görtler vortices is evaluated for compressible boundary-layer flows, and the effect of the growth of the boundary layer is included in the analysis.

Dimensionless generalized field equations are expressed in a coordinate system (x,y,z) based on the streamlines and potential lines of the inviscid flow over a weakly curved surface. Three-dimensional, steady, spatially growing disturbances which are periodic in the Cartesian coordinate z are superposed on each basic flow quantity as

$$(\hat{u},\hat{v},\hat{p},\hat{\theta}) = (U,V,P,\Theta) + (u,v,p,\theta)\cos(\beta z)\exp(\textstyle\int \sigma \mathrm{d}x)$$

$$\hat{w} = 0 + w(y)\sin(\beta z)\exp(\textstyle\int \sigma \mathrm{d}x) \tag{1}$$

where U, V, P, and Θ are the basic flow velocities, pressure, and temperature, respectively; u, v, w, p, and θ are the corresponding disturbance quantities; and β and σ are the wavenumber and growth rate of the disturbance. The field equations are made dimensionless by introducing the length scale $L^* = \sqrt{\nu^*_\infty x^*/U^*_\infty}$ for the y and z coordinates; the velocity scale U^*_∞ for U and u; the scale RU^*_∞ for V, v, and w; and the scale $R^2\rho^* U^{*2}_\infty$ for the pressure disturbance. With this scaling the distrubance motion and basic flow quantities change in the x coordinate according to a new scale $x = x^*/RL^*$, where R is the Reynolds number based on U^*_∞ and L^*. By substituting Eqs. (1) into the field equations, subtracting the basic flow, linearizing the equations, and keeping the leading order terms, we obtain the following disturbance equations.

$$\left[\frac{1}{\Theta}(\sigma U + U_x) + \mu\beta^2\right]u + \left(\frac{V}{\Theta} - \mu_y\right)u_y - \mu u_{yy} + \frac{1}{\Theta}U_y v$$
$$- \left[\frac{1}{\Theta^2}(UU_x + VU_y) + (\tilde{\mu}U_y)_y\right]\theta - \tilde{\mu}U_y\theta_y = 0 \tag{2}$$

$$\left[\frac{1}{\Theta}(V_x + 2UG^2) - c\sigma\mu_y\right]u - [(c+1)\sigma\mu + \mu_x]u_y$$
$$+ \frac{1}{\Theta}(V_y + \sigma U + \mu\beta^2)v + \left[\frac{V}{\Theta} - (c+2)\mu_y\right]v_y$$
$$- (c+2)\mu v_{yy} - c\mu_y\beta w - (c+1)\mu\beta w_y + p_y$$
$$- \left[\frac{1}{\Theta^2}(UV_x + VV_y + U^2G^2) + (c+1)\tilde{\mu}U_{xy} + c\tilde{\mu}_y U_x\right.$$
$$\left. + (c+2)(\tilde{\mu}V_y)_y + \tilde{\mu}_x U_y + \sigma\tilde{\mu}U_y\right]\theta$$
$$- [c\tilde{\mu}U_x + (c+2)\tilde{\mu}V_y]\theta_y = 0 \tag{3}$$

$$\beta[(c+1)\sigma\mu + \mu_x]u + \beta\mu_y v + (c+1)\beta\mu v_y$$
$$+ \left[(c+2)\beta^2\mu + \frac{\sigma U}{\Theta}\right]w - \left(\mu_y - \frac{V}{\Theta}\right)w_y - \mu w_{yy}$$
$$- \beta p + c\beta\tilde{\mu}(U_x + V_y)\theta = 0 \tag{4}$$

$$\frac{1}{\Theta}\left(\sigma - \frac{1}{\Theta}\Theta_x\right)u - \frac{1}{\Theta^2}\Theta_y v + \frac{1}{\Theta}v_y + \frac{\beta}{\Theta}w + \frac{1}{\Theta^2}\left(\frac{2U}{\Theta}\Theta_x\right.$$
$$\left. + \frac{2V}{\Theta}\Theta_y - U_x - V_y - \sigma U\right)\theta - \frac{V}{\Theta^2}\theta_y = 0 \tag{5}$$

$$\frac{1}{\Theta}\Theta_x u - 2(\gamma-1)M^2_\infty \mu U_y u_y + \frac{1}{\Theta}\Theta_y v - \left[\frac{1}{\Theta^2}(U\Theta_x + V\Theta_y)\right.$$
$$\left. + (\gamma-1)M^2_\infty\tilde{\mu}(U_y)^2 + \frac{1}{\Gamma}\mu_{yy} - \frac{\sigma U}{\Theta} + \frac{\beta^2\mu}{\Gamma}\right]\theta$$
$$+ \left(\frac{V}{\Theta} - \frac{2}{\Gamma}\mu_y\right)\theta_y - \frac{1}{\Gamma}\mu\theta_{yy} = 0 \tag{6}$$

where G is the Görtler number defined by $G = R\sqrt{L^*/r^*}$, and r^* the radius of curvature of the streamline at $y=0$. Moreover, M_∞ is the freestream Mach number, Γ Prandtl number, μ the basic flow viscosity, $\tilde{\mu} = \mathrm{d}\mu/\mathrm{d}\Theta$, $c = 2(e-1)/3$, and $e = 0.8$. Subscripts x, y, and z denote $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, respectively. Equations (2-6) represent the x-, y-, z-momentum, mass, and energy equations, respectively. The

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*Assistant Professor, Dept. of Mechanical Engineering and Mechanics; presently with King Abdulaziz University, Dept. of Engineering Maths, Jeddah, Saudi Arabia. Member AIAA.

†Graduate Student.

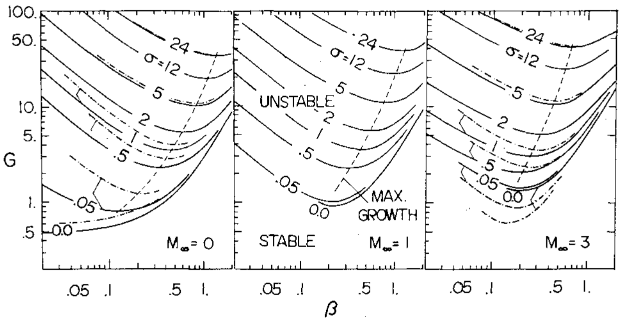

Fig. 1 Contours of constant growth rates at Mach numbers of 0, 1, and 3. Terms due to boundary-layer growth included, ____; terms due to boundary-layer growth excluded, -.-.-.-.

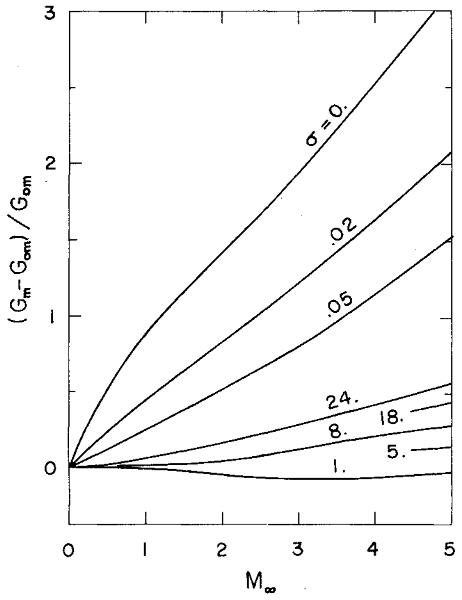

Fig. 2 Effect of compressibility on local stability.

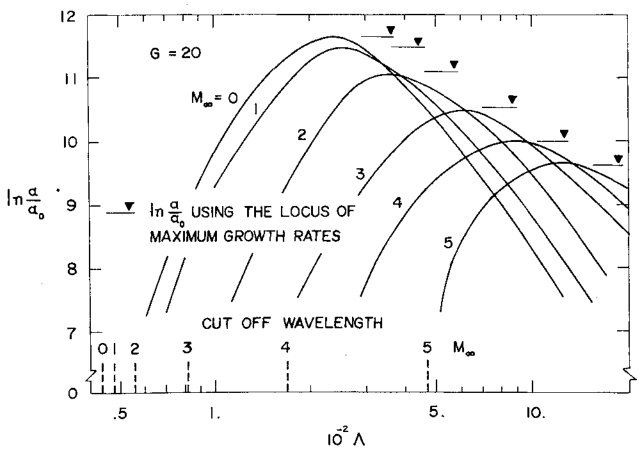

Fig. 3 Effect of compressibility on the maximum amplitude ratio calculated along a growth path of constant wavelength Λ.

boundary conditions are

$$u = v = w = \theta_y = 0 \qquad \text{at} \quad y = 0 \tag{7}$$

$$u, v, w, \theta \to 0 \qquad \text{at} \quad y \to \infty \tag{8}$$

Equations (2-8) constitute an eighth-order system of homogeneous linear ordinary differential equations that form an eigenvalue problem for the parameters β, σ, and G.

Figure 1 shows contours of constant growth rates plotted in the G-β plane for $M_\infty = 0$, 1, and 3. Instability of boundary-layer flows sets in at higher Görtler numbers as M_∞ increases for all wavenumbers higher than 0.1. The critical value of Görtler number G_c, below which the flow is stable for any disturbance wavenumber, increases as M_∞ increases. Growth rate curves possess minima (G_m) that form a locus of maximum growth rates of different wavenumber components. As M_∞ increases these minima occur at higher Görtler numbers and lower wavenumbers. Figure 1 shows curves of constant growth rates calculated by excluding all terms due to boundary-layer growth. These terms greatly influence the region near the neutral curve, where they have opposite effects for $M_\infty = 0$ and 3.

The parameter $(G_m - G_{0m})/G_{0m}$ is used in Fig. 2 as an ordinate to indicate the stabilizing effect (increase in G_m) as M_∞ increases compared with G_{0m}, the minimum Görtler number at $M_\infty = 0$, for the corresponding value of growth rate σ. The figure shows that compressibility has its maximum stabilizing influence when the vortex is weak (low growth rate).

Let a_0 and G_0 be the amplitude of the vortex and Görtler number at the beginning of the unstable region. The amplitude ratio of the vortex at any downstream location is calculated using $\ell n(a/a_0) = (4/3)\int(\sigma/G)\mathrm{d}G$. The integration is carried from G_0 *to* $G = 20$ (for comparison purposes) along a growth path that conserves the disturbance dimensional wavelength λ^* that is along a constant dimensionless parameter $\Lambda = (U^*_\infty \lambda^* / \nu^*_\infty)\sqrt{\lambda^*/r^*}$. Results are shown in Fig. 3 for different Mach numbers. As M_∞ increases there is a reduction in the maximum value of the amplitude ratio of the vortices, as well as a shift of the most unstable wavelength Λ to a higher value. Figure 3 also shows an increase in the upper band of the disturbance wavelength that is always attenuated (cutoff wavelength) as M_∞ increases. It is worth noting that the values of $\ell n(a/a_0)$ at $G = 20$ calculated by integrating along the path of maximum growth rates are almost identical to the corresponding values of $\ell n(a/a_0)$ at the most unstable wavelength for different Mach numbers.

Reference 1 contains a detailed account of the analysis and numerical procedures, as well as a literature review. Also discussed in Ref. 1 is the effect of compressibility on the shape of the eigenfunctions.

Acknowledgment

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References

[1] El-Hady, N. M. and Verma, A. K., "Growth of Görtler Vortices in Compressible Boundary Layers Along Curved Surfaces," AIAA Paper 81-1278, June 1981.